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Lattice-Theoretic Foundations of the Consumer's Problem

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Abstract

This paper provides an introduction to and discussion of the application of lattice-theoretic methods to classic problems in consumer theory. General characterizations of income effects with two goods, and with an arbitrary number of goods, as well as examples of comparative statics over densities and consumer types are also presented.

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1 Introduction

The theory of the consumer has a long and venerable history in economic science. It rests on two components: the existence of an optimum, and the resulting comparative statics properties. That is, first, given a set of indifference curves that are differentiable and strictly convex to the origin, there is a unique optimizer. Second, this optimal point varies with parameters in a way that can be characterized. For example, the optimal choice varies with the price of a good, thereby describing a demand curve. Also, a sufficient condition for a good's demand curve to be downward-sloping is that it be normal, i.e. that the optimal choice be increasing in income.

The pioneers of modern demand theory, Hicks, Samuelson, and Slutsky, worked against the backdrop of Marshall's contributions, which assumes decreasing marginal utility as a basis for demand curves. Although the received intuition as to comparative statics may have been correct, they found it necessary to ground the theory on a "surer foundation", which they did by replacing the law of diminishing marginal utility with the principle of diminishing marginal substitution.[1] This led to the common assumptions of differentiability and strict quasiconcavity which are technically necessary in order to obtain the interior optimum solution to which the implicit function theorem could be applied. These assumptions have since been attacked. It is generally accepted that on the one hand, these assumptions on preferences appear to be more stringent than is necessary in order to derive demand curves, yet, on the other, they are the ones that have yielded the well-known comparative statics that are the basis of consumer theory.

In this paper, we study several comparative statics that arise from the consumer's problem, by means of the lattice programming techniques developed in Veinott [1992] (see also Li Calzi and Veinott [1992]) and applied in Antoniadou [1995].[2] Their approach rests upon the notion of a lattice space, which consists

[1] Hicks' justification is utilitarian in nature (the "oddities" he refers to below are multiple optimum solutions):

> We can, I think, get a surer foundation if we reflect on the purpose for which we require our principle. We want to deduce from it laws of market conduct ... The general principle of diminshing marginal substitution merely rules out these oddities; by that principle we select the simplest of the various possibilities among us. (Hicks 1939, p. 23)

[2] Mirman, Morand and Reffett [2002] apply the methods employed here to macroeconomic

of a set and an appropriate associated partial order. Because lattice properties are independent of topological ones, the approach dispenses with the standard differentiability and quasiconcavity assumptions. Instead, the comparative statics of optimizers are characterized for functions that are lattice superextremal. This property corresponds to, loosely speaking, a form of complementarity in a function's arguments and is independent of differentiability and concavity. The comparative statics results obtain over constraint sets that "increase" in a set ordering, the strong set order. For this condition to hold as a parameter varies, the partial order associated with the lattice space must be chosen to be consistent with the application. Otherwise, the theorem does not yield a meaningful result, which is often the case when the partial order is the customary Euclidean order.

Our focus is on comparative statics only. In contrast with the historical development outlined above, the problem of the existence of optimal solutions is relegated to a more ancillary role, as there are other methods with which to address it. In fact, the lattice theoretic approach allows it to be a distinct issue. The backbone of the lattice theoretic approach that we study in this paper is to define a decision space and, then, to find an appropriate order on this space, making it a lattice space, so that the theorem of Veinott can be applied. Although we introduce and make use of several nonstandard partial orders, these serve only in characterizing the comparative statics of optimal choices and do not affect the underlying maximization problem.

Basically, two ingredients are necessary. First the constraint set must be strongly ordered in the partial order and, second, the objective function must be lattice superextremal. It is shown, for example, that the Euclidean order does not accomplish this task. In particular, budget sets are not strongly ordered by the Euclidean order and, thus, the theorem does not apply. Antoniadou introduces an order to replace the Euclidean order, this is the direct value order.

This direct value order combines the value along a budget constraint line and the lexicographic order, yielding budget sets that increase in the strong set order as income increases. It then follows that Veinott's central characterization theorem applies for functions that are lattice superextremal in this value order. Therefore, a sufficient condition for the optimal choice (set) to increase, and,

models.

therefore, the good under consideration to be normal, is that the utility function be lattice superextremal in this value order. We explore the ability of the lattice theoretic approach to predict whether the optimal choice is normal for different classes of preferences when there are two goods. A first result is that the lattice approach overlaps with the traditional one if the utility function is differentiable and strictly quasiconcave. It also provides a weaker condition for normality if the good is required to be normal for a specific price only, assuming that the utility function is differentiable. In addition, using several examples when preferences are either not differentiable (like Leontief preferences), or not strictly quasiconcave, we show how, when points are ordered by the direct value order, these examples fall within the scope of Veinott's characterization theorem.

When the utility function is not strictly quasiconcave, the income expansion path may not only be a correspondence, it may, in addition, not be convex-valued. In this case, the Antoniadou value order is not sufficient to invoke Veinott's theorem. We introduce an alternative partial order that is consistent with income effects in the nonquasiconcave case, the radial value order. This radial value order is a partial order that ranks budget sets in the strong order. We then show, again by example, how Veinott's theorem can be invoked to get monotonic optimizers, by showing that the objective functions (characterized by nonquasiconcavity) are indeed lattice superextremal in this radial order. This leads to the result that for all homothetic utility functions both goods are normal for all prices. In contrast with the customary approach, which uses Roy's identity, neither differentiability nor quasiconcavity are imposed.

Applications of the characterization theorem, thus, hinge on the partial order underlying the lattice space. The lattice theoretic approach is equally adapted to comparative statics in infinite dimensional spaces so long as these are partially ordered. Such spaces arise if the parameter that varies is a function rather than a scalar. This is illustrated for a shift in preferences. In this case, the characterization theorem establishes that a single-crossing condition suffices to determine whether consumption of a good increases as the utility function varies. As an application to decision-making in a stochastic environment, we study an optimal investment problem and derive the condition that determines whether investment increases as the asset becomes more risky. In the final example of the paper, it is the choice space rather than the parameter space that is infinite dimensional. In order to extend the analysis to an infinite dimensional choice space we generalize the notion of a direct value order in two dimensions to an arbitrary dimensional space. This is the canonical value order. Applying the characterization in this environment establishes a final economic result. We show that for a separable utility function only strict monotonicity of each good is needed for all goods to be normal at any price.

The purpose of the exercises in this paper is to illustrate the usefulness of the lattice theory approach. We emphasize that the method is very powerful. It yields results, for the example, when the objective function is not differentiable and not strictly quasiconcave, as well as more general situations,e.g., the objective function and the constraint sets are interrelated. The major point is that to apply lattice theory techniques, it is necessary to find the right space over which an appropriate partial order is applied, i.e., for the constraint sets to be strongly ordered and the objective function to be lattice superextremal. Indeed, there are many problems that can be solved with these techniques when the right space and the right partial order can be found. .

2 Mathematical Background

The lattice-theoretic approach rests upon the notion of a lattice space. A lattice space is defined over a pair $[X, \geq_X]$, where X is a set and $\geq_X$ is a partial order on X.[3] The partial order $\geq_X$ defines the binary operations $\vee$ and $\wedge$ which denote the least upper bound and the greatest lower bound, respectively, of any two points in X (these are also referred to as the "join" and the "meet"). A lattice space is a pair $[X, \geq_X]$ that is closed under $\vee$ and $\wedge$, that is, $x \vee y$ and $x \wedge y$ exist and are elements of X. Although "lattice space" is a lengthier locution than "lattice", this terminology has the advantage of emphasizing the fact that a lattice space has two components, a set and a partial order, which avoids the temptation to identify a lattice with the underlying set X. To fix ideas, consider the following:

Example 1 *Let $X = [a_1, a_2] \times [b_1, b_2]$ with the Euclidean order, $\geq_\varepsilon$.*[4] *Then, $[X, \geq_\varepsilon]$ is a lattice space. $[X', \geq_\varepsilon]$ with $X' = (a_1, a_2) \times (b_1, b_2)$ is also a lattice space since $x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}) \in X'$ and $x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}) \in X'$.*

The next two examples are not lattice spaces. The first uses the Euclidean order but over a different set X. In the second, the set X is as in Example 1 but the order is different.

Example 2 *Let $X = \left\{(x_1, x_2) \in \Re^2_+ \text{ s.t. } p_1 x_1 + p_2 x_2 \leq I\right\}$ be endowed with the Euclidean order. Then $(I/p_1, 0) \vee (0, I/p_2) = (I/p_1, I/p_2) \notin X$, so $[X, \geq_\varepsilon]$ is not a lattice space.*

Example 3 *Let $X = [0, 1] \times [0, 1]$ but now order points by their polar coordinates. Suppose that x and y have the Euclidean coordinates $(1, 1)$ and $(0, 1)$, respectively. Then, using the order $\geq_{\rho,\theta}$ to define the join operation, $x \vee y = (0, \sqrt{2}) \notin X$, so $[X, \geq_{\rho,\theta}]$ is not a lattice space.*

Since budget sets arise frequently in economics, Example 2 seems to limit the applicability of lattice-theoretic methods in some important cases. The next example shows that any subset of $\Re^2_+$ can be made into a lattice space in a trivial

[3] A partial order is reflexive ($x \geq x$), antisymmetric ($x \geq y$ and $y \geq x \Rightarrow x = y$), and transitive ($x \geq y$ and $y \geq z \Rightarrow x \geq z$).

[4] In $\Re^n$ with $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, $x \geq_\varepsilon y$ if $x_i \geq y_i$ for $i = 1, ..., n$.

sense, in that there are no incomparable points. As shown in section 3 below, budget sets can be embedded into other lattice spaces that are more useful in applications.

Example 4 *Let X be any subset of $\Re^2_+$ and take the lexicographic order $\geq_l$ with $x = (x_1, x_2) \geq_l (y_1, y_2) = y$ if $x_1 > y_1$, or $x_1 = y_1$ and $x_2 \geq y_2$. Then, $[X, \geq_l]$ is a lattice space: $x \vee y$ is equal to either x or y, and similarly for $x \wedge y$.*

These examples are stated for $\Re^2_+$, but carry over directly to $\Re^n_+$. The theorems presented below also apply to infinite-dimensional spaces, as long as these can be partially ordered (by stochastic dominance, for example) in a way that induces a lattice space. Thus, the partial order imposed on the underlying set X plays a key role in determining its lattice-theoretic properties. Moreover defining lattice spaces often involves partial orders other than the common component-wise Euclidean one.

Three more definitions are necessary in order to state the main lattice-theoretic comparative static results. The first is that of a *lattice superextremal* (LSE) function:[5]

Definition 5 *$f : [X, \geq_X] \to \Re$ is lattice superextremal (LSE) if, for all x and y in X,*

$$f(x) \vee f(y) \leq f(x \wedge y) \vee f(x \vee y) \text{ or } f(x) \wedge f(y) \leq f(x \wedge y) \wedge f(x \vee y),$$

$$\text{and } f(x) = f(x \vee y) \Longrightarrow f(y) \leq f(x \wedge y),\ f(x) = f(x \wedge y) \Longrightarrow f(y) \leq f(x \vee y)$$

The LSE property holds trivially for any pair of comparable points x and y, i.e. such that $x \wedge y$ and $x \vee y$ are equal to either x or y. It therefore only needs to be checked for pairs of incomparable points. The following characterization of the LSE property is often used in practice:

Proposition 6 *(Veinott) f is LSE if and only if for all x and y in X,*

$$f(x) \underset{(>)}{\geq} f(x \wedge y) \Rightarrow f(x \vee y) \underset{(>)}{\geq} f(y)$$

Although the partial order $\geq_X$ that determines the lattice space does not appear explicitly in Proposition 6, it serves to determine the join and meet operations that do appear. In the case that $[X, \geq_X] = [\Re^n_+, \geq_\varepsilon]$, the intuition

[5] There are five variants of the superextremal property of which lattice superextremality and strict superextremality (see Appendix) appear to be the more useful. An LSE function is *quasisupermodular* in the terminology of Milgrom and Shannon [1994].

behind the LSE property is expressed as follows. The change from $x \wedge y$ to x corresponds to an increase in some of the components of $x \wedge y$. The characterization in the proposition states that if this change results in a higher value of f, then it also results in a higher value of f if the other components of $x \wedge y$ are increased. The LSE property is thus said, loosely, to correspond to a type of complementarity between the different components. The LSE property can be contrasted with another lattice-theoretic property, supermodularity.[6] LSE is the weaker of the two. For example, the LSE property is ordinal (preserved under strictly increasing monotonic transformations), while supermodularity is not.

The second definition is the *single-crossing property* (SCP):

Definition 7 $f : X \times T \longrightarrow \Re$ *satisfies the single-crossing property (SCP) if, for* $\overline{x} \geq_X \underline{x}$ *and* $\overline{t} \geq_T \underline{t}$, $f(\overline{x},\underline{t}) \geq f(\underline{x},\underline{t}) \Longrightarrow f(\overline{x},\overline{t}) \geq f(\underline{x},\overline{t})$ *and* $f(\overline{x},\underline{t}) > f(\underline{x},\underline{t}) \Longrightarrow f(\overline{x},\overline{t}) > f(\underline{x},\overline{t})$.

$[X, \geq_X]$ needn't be a lattice space in order for f to satisfy the SCP since the definition does not involve joins and meets in X. If two points x and y in X are incomparable, the implications needn't be verified and the SCP property is trivially satisfied. In order to relate the SCP to the LSE property, consider the restriction of f to lattice spaces over $\widetilde{X} = (\overline{x}, \underline{x}) \times (\overline{t}, \underline{t})$ for given $\overline{x} \geq_X \underline{x}$ and $\overline{t} \geq_T \underline{t}$, with $\widetilde{X}$ ordered by the componentwise order. If f is LSE over $\left[\widetilde{X}, \geq_{\widetilde{X}}\right]$, then f satisfies the SCP. Therefore, the SCP is weaker than the LSE property.[7]

The third necessary notion is that of what it means for a set to increase. This is done by means of the *strong set order* $\geq_S$, which is defined as:

Definition 8 $B \geq_S A$ *if for all* $a \in A$ *and* $b \in B$, $a \wedge b \in A$ *and* $a \vee b \in B$.

[6] A function $f : X \to \Re$ is supermodular if $f(x) + f(y) \leq f(x \wedge y) + f(x \vee y)$.

[7] There are two incomparable points, $x = (\overline{x}, \underline{t})$ and $y = (\underline{x}, \overline{t})$. The meet and join are $x \wedge y = (\underline{x}, \underline{t})$ and $x \vee y = (\overline{x}, \overline{t})$. If f is LSE, then:

$$\begin{cases} f(x) = f(\overline{x},\underline{t}) \underset{(>)}{\geq} f(\underline{x},\underline{t}) = f(x \wedge y) \\ \Longrightarrow f(x \vee y) = f(\overline{x},\overline{t}) \underset{(>)}{\geq} f(\underline{x},\overline{t}) = f(y), \\ f(y) = f(\underline{x},\overline{t}) \underset{(>)}{\geq} f(\underline{x},\underline{t}) = f(x \wedge y) \\ \Longrightarrow f(x \vee y) = f(\overline{x},\overline{t}) \underset{(>)}{\geq} f(\overline{x},\underline{t}) = f(x) \end{cases}$$

The first of these implications is the SCP condition.

As an example, $f(0,0) = 1$, $f(1,0) = 4$, $f(0,1) = 2$, and $f(1,1) = 3$ satisfies the SCP (with t being the second component) but is not LSE.

Taking $B \geq_S A$, it is possible to characterize the strong set order by means of selections $\{a, b\}$ from the Cartesian product $A \times B$. The points in a given selection are not necessarily comparable. The definition does guarantee the existence of at least one weakly increasing selection, $\{a \wedge b, a \vee b\}$. Also, suppose that there is a strictly decreasing selection $\{a, b\}$. Then, $a \wedge b = b \in A$ and $a \vee b = a \in B$, so $\{b, a\}$ is also a selection from $A \times B$: any decreasing selections therefore must lie in $A \cap B$.

Two related theorems characterize comparative statics in lattice spaces.[8] We cite an abbreviated version of Veinott's formulation[9] that we refer to hereafter as (V):

Theorem 9 *(Veinott) $f : [X, \geq_X] \to \Re$ is LSE if and only if $B^* = \arg\max_{x \in B} f(x) \geq_S \arg\max_{x \in A} f(x) = A^*$ for all A, $B \subseteq X$ with $B \geq_S A$ and $\arg\max_{x \in A} f(x)$,$\arg\max_{x \in B} f(x) \neq \emptyset$.*

A and B can be any pair of sets that is ranked by the strong set order. The fact that $[X, \geq_X]$ is a lattice space plays a key role, since the order $\geq_X$ defines the meet and join operations that ultimately determine both whether f is LSE and whether the sets B and A are ranked in the strong set order. The theorem establishes an equivalence relationship between a global property of the objective function, and the direction of the comparative statics over all pairs of strongly comparable sets.

Because f is not assumed to be strictly quasiconcave, the set of optimal points need not be a singleton. In this case, the theorem asserts that the sets of optimal points are increasing in the strong set order. This does not imply that optimum points increase in the conventional sense. Rather, as discussed above, this allows for selections consisting of incomparable points, but guarantees that there exists a weakly increasing selection from the optimal sets, and that decreasing selections lie in $A^* \cap B^*$. When the sets of optimal points are singletons, then the theorem asserts that the constrained maximum is nondecreasing.

(V) applies when the comparative statics can be cast in terms of variations in a constraint set. Milgrom and Shannon [1994] provide an alternative formulation of which (V) is a special case. The principal difference is that the comparative

[8] See Antoniadou [1994], p. 25 and Li Calzi [1992], p. 2.

[9] Veinott's actual formluation applies to all five variants of the superextremal property of which the LSE property is one instance. In Li Calzi [1992], the equivalence extends to other properties besides the LSE and comparative static properties mentioned here.

statics can be either a variation in the constraint set, or variation in a parameter t that enters into the function and eventually into the constraint set. We refer to this formulation, hereafter, as (MS).

Theorem 10 *(Milgrom and Shannon) Let* $f : X \times T \longrightarrow \Re$ *where* $[X, \geq_X]$ *is a lattice space and* T *is a partially ordered set.* $\arg\max_{x \in S \subseteq X} f(x,t)$ *is nondecreasing in* (t, S) *if and only if* f *is LSE in* x *and satisfies the SCP in* $(x;t)$.

If t is constant and only S varies, (V) and (MS) are identical. The difference resides in the presence of the parameter t that is treated differently from the choice variable x. (MS) does not require f to be LSE in all its arguments. Rather, it imposes the less stringent requirements that f be LSE in its x argument only and satisfy the weaker SCP property with regard to t. This suggests that (MS) can characterize comparative statics in some applications where (V) cannot.

The function f given at the end of footnote 7 above illustrates this difference. (V) cannot be used to assert that $\arg\max_{t=t^0} f$ is increasing in t because f is not LSE in the Euclidean order $\geq_\varepsilon$. However, f is LSE in x, trivially, since x is a scalar and satisfies the SCP in $(x;t)$ so (MS) can be applied. Note that $\arg\max_{t=t^0} f$ is increasing in t, but if one were to treat t as the choice variable and x as the parameter, then $\arg\max_{x=x^0} f$ is not increasing in x. Taking $B = \{1\} \times \{0,1\} \geq_S \{0\} \times \{0,1\} = A$, the maximizer over A is (1,0) (and (0,1) over B), so $B^* \ngeq_S A^*$. Since there is a pair of strongly increasing constraint sets over which the maximizers are not increasing, f should not be expected to be LSE in both its arguments. (MS) weakens the LSE requirement by imposing the restriction (which A and B violate) that constraint sets be of the form $S \times \{t\}$.

The choice between the (V) (which emphasizes strongly ordered constraint sets) and the (MS) (which emphasizes the SCP) formulations hinges on whether the application is conceived in terms of a variation in a constraint set or in terms of a variation in a parameter in the objective function. These cases are not mutually exclusive.

The above definitions and theorems constitute the principal lattice-theoretic tools for comparative statics. The thrust of the technique lies in its use of order-based properties, rather than topological ones of which it is independent. The lattice-theoretic approach does not therefore require quasiconcavity, differentiability of even continuity assumptions, and applies to infinite-dimensional spaces much as it applies to finite-dimensional ones, so long as they can be partially ordered to form a lattice space. However, in order to gain familiarity with the lattice-theoretic properties, once sufficient conditions for comparative statics have been devised in the examples that follow, differentiability will be often assumed in order to compare the resulting conditions to the ones that result from the standard approach.

3 Income Effects

In this section, we study conditions on preferences under which comparative static results can be derived for the consumer's problem using (V).[10] The first four subsections discuss income effects when there are two goods, and the fifth subsection extends these results to the case of countably many goods. The focus throughout is on whether optimal consumption of good 1 increases for the preferences represented by the utility function $U(x)$ at prices p as income increases from $\underline{I}$ to $\overline{I}$.

The lattice-theoretic approach allows for multiple optimal solutions. In this context, the notion of increasing consumption (that is of good 1 being normal) needs to be specified. The following definition is adapted to this framework. It allows for the possibility that there may be decreasing selections from the optimal set as income increases, but guarantees the existence of at least one weakly increasing selection.

Definition 11 *A good is normal if every optimal choice at high income is weakly greater than some optimal choice at low income and every optimal choice at low income is weakly smaller than some optimal choice at high income.*

3.1 The Direct Value Order

The objective is to derive a sufficient condition for the utility maximizing choice (set) to be increasing by applying (V) to the budget sets $A = \left\{x \in \Re^2_+, \text{ s.t. } p \cdot x \leq \underline{I}\right\}$ and $B = \left\{x \in \Re^2_+, \text{ s.t. } p \cdot x \leq \overline{I}\right\}$. Two conditions must be fulfilled in order for (V) to characterize the behavior of the set of optimizers. First, the function U must be LSE and, second, the budget sets A and B must ranked by the strong set order. Both of these conditions depend on the underlying partial order imposed on $\Re^2_+$.

Consider first the Euclidean order. Let $a = (0, \underline{I}/p_2) \in A$ and $b = (\overline{I}/p_1, 0) \in B$. Then, given the join operation that $\geq_\varepsilon$ induces, $a \vee b = (\overline{I}/p_1, \underline{I}/p_2) \notin B$. Therefore, if the lattice space is $\left[\Re^2_+, \geq_\varepsilon\right]$ the budget sets A and B are not

[10] The (MS) formulation applies when there are two goods, by substituting the budget constraint into the objective function. With two goods, this reduces the problem to a single choice variable so U is trivially LSE, and the SCP is directly verified. However, this approach runs into difficulties, similar to those discussed in section 3.1, when there are three or more goods. In this case, the (V) formulation allows for a more informative exposition, for instance of the underlying geometry.

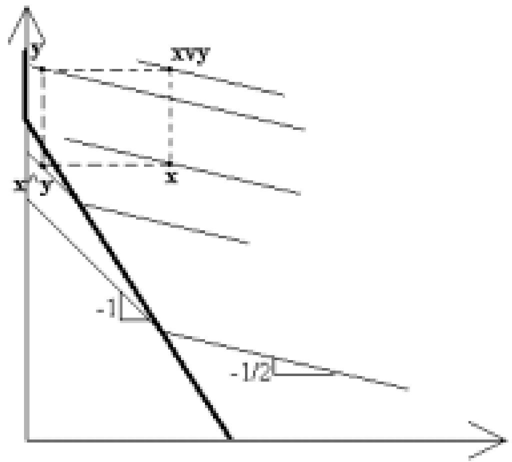

Figure 1:

increasing in the strong set order sense, i.e. $B \ngeq_S A$ and (V) cannot be invoked for this comparative statics problem.

To illustrate that the theorem does not apply in this case, take the preferences depicted in Figure 1:

Example 12 *Suppose that* $U(x) = \begin{cases} x_1 + x_2 \text{ for } 2x_1 + x_2 \leq 3 \\ \frac{x_1+2x_2}{3} + 1, \text{ for } 2x_1 + x_2 \geq 3 \end{cases}$ *. For all* $x, y \in \Re^2_+$, $U(x) < U(x \wedge y)$ *but* $U(x \vee y) > U(y)$ *(see Figure 1). Therefore, the implications in Proposition 6 hold and* $U(x)$ *is LSE in the Euclidean order.*

If the relative price is $p_1/p_2 \in (1/2, 1)$, the optimal choice x_1^* decreases as income increases. The income expansion path is the bold line in the figure. Thus, there are prices for which $U(x)$ is LSE in the Euclidean order but the optimal set decreases as income increases.[11]

To apply (V) it is, therefore, necessary to use a partial order on $\Re^2_+$, that is suitable for income effects, that is, under which the low and high budget sets are ranked in the strong set order. Antoniadou [1995] solves this problem of budget set comparability by introducing the *direct value order* $\geq_{d.v.(p)}$. It is

[11] For any objective funtion f whose level sets have monotone slopes that are neither zero nor infinite, the Euclidean order is unlikely to be of great use. If the level sets are decreasing for example, then $f(x \vee y) \geq f(x)$ for all x and y in $\Re^2$. Therefore, any such function is LSE $\left[\Re^2_+, \geq_\varepsilon\right]$, so the LSE conditions do not provide an informative restriction for comparative statics.

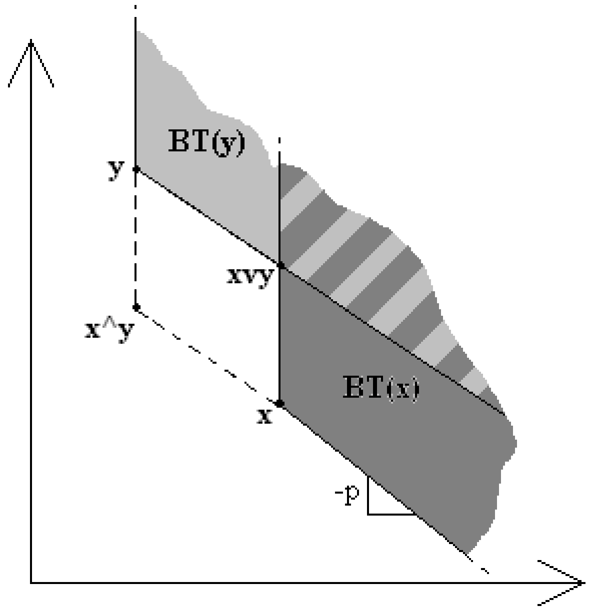

Figure 2:

defined for a given vector of prices p and given by:[12]

Definition 13 $x \geq_{d.v.(p)} y$ *if* $p \cdot x \geq p \cdot y$ *and* $x_1 \geq y_1$

In what follows, we also use p to refer to the magnitude of the slope associated with the value order, p_1/p_2. The direct value order is characterized by the weakly bigger than sets $BT(x) = \left\{z \in X \text{ s.t. } z \geq_{d.v.(p)} x\right\}$. These are depicted in Figure 2, which also shows the resulting join and meet for two incomparable points x and y. The coordinates of the meet and join in the Euclidean order, taking $x_1 \geq y_1$ without loss of generality, are $x \wedge y = (y_1, x_2 + p(x_1 - y_1))$ and $x \vee y = (x_1, y_2 - p(x_1 - y_1))$. Figure 2 indicates that $x \vee y$ is the smallest (in the direct value order) among those points that are bigger than both x and y, i.e. those that lie in the striped region.

The direct value order has two properties that are useful for income effects. First, it discriminates between those cases where good 1 is normal and those

[12] Antoniadou's direct value order is defined over $\Re^n$ as $x \geq_{d.v.(p)} y$ if $p \cdot x \geq p \cdot y$ and $x \geq_l y$ where $\geq_l$ is a lexicographic order. For $n > 2$ joins can have a negative component and $\left[\Re^n_+, \geq_{d.v.(p)}\right]$ is not a lattice space. The Euclidean order corresponds to the direct value order for $p_1 = 0$.

where it is not. Suppose that (V) applies, so the optimal choice set is increasing in the order $\geq_{d.v.(p)}$ for some price p. Then for any optimal choice x^* there is an income expansion path that lies in x^*'s bigger than set along which x_1^* increases weakly. As long as $U(x)$ is increasing in the Euclidean order, so both goods are desirable, points to the right of x^* but below the budget line are never optimal at higher incomes. Therefore, the better than sets defined by the direct value order do not rule out any income expansion paths for which good 1 is normal.

The direct value order also solves the problem of budget constraint comparability. In the lattice space $\left[\Re_+^2, \geq_{d.v.(p)}\right]$, budget constraints are ranked in the induced strong set order as long as the slope p associated with the partial order is the same as that of the budget constraint. In contrast with the example above where meets and joins were defined by the Euclidean order, the join $x \vee y$ for any pair of points does not lie outside the set B under the direct value order.[13] Thus, for a function $U(x)$ that is LSE in the order $\geq_{d.v.(p)}$, since $B \geq_S A$, the set of maximizers of U is increasing in the strong set order as income increases from $\underline{I}$ to $\overline{I}$ which implies that good 1 is normal.[14]

3.2 The LSE Property in $\geq_{d.v.(p)}$ for U Quasiconcave

Since the direct value order $\geq_{d.v.(p)}$allows (V) to be applied, characterizing normality is the same as characterizing functions that are LSE in $\geq_{d.v.(p)}$. To see what this property entails, we begin with three examples of functions that are LSE: the utility functions representing Cobb-Douglas preferences, Leontief preferences, and a generalization of Leontief preferences. Only the first of these is differentiable. Furthermore, the last example is chosen to be quasiconcave, but not strictly quasiconcave. As a result, there is a relative price for which the expansion path is a correspondence.

To establish whether a function $U(x)$ is LSE in the order $\geq_{d.v.(p)}$, it suffices to verify that the implications stated in Proposition 6 hold for any pair of incomparable points x and y, using the meet $x \wedge y$ and join $x \vee y$ defined by the direct value order. The LSE property holds trivially for pairs of comparable points.

[13] If the slope of the budget constraint differs from the one that defines the value order, then this strong set comparability fails to hold. For example, if the slope of the value order is smaller than that of the budget set in magnitude, then it is possible that $x \vee y \notin B$.

[14] Changing the order on $\Re_+^2$, the domain of U, does not affect the maximization problem and the optimum set.

Example 14 *Suppose that* $U(x) = x_1x_2$*. Let* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *be two incomparable points such that* $x_1 \geq y_1$ *and that* $y_2 \geq x_2$*. Express the meet and join as* $x \wedge y = (y_1, y_2 - h)$ *and* $x \vee y = (x_1, x_2 + h)$ *where* $h = (y_2 - x_2) - p(x_1 - y_1)$*. There are two possible types of violations of the condition in Proposition 6:*[15] $\{U(x) \geq U(x \wedge y)$ *and* $U(y) > U(x \vee y)\}$ *or* $\{U(y) \geq U(x \wedge y)$ *and* $U(x) > U(x \vee y)\}$*. Consider the first:*

$$U(x) = x_1x_2 \geq y_1(y_2 - h) = U(x \wedge y)$$
$$\Rightarrow x_1x_2 + hy_1 \geq y_1y_2$$
$$\Rightarrow U(x \vee y) = x_1(x_2 + h) \geq y_1y_2 = U(y).$$

The second holds trivially since, although $U(y) = y_1y_2 \geq y_1(y_2 - h) = U(x \wedge y)$*,* $U(x \vee y) = U(x_1, x_2 + h) \geq U(x_1, x_2) = U(x)$*. Therefore,* U *is LSE in the direct value order.*

This function is LSE for all p and the utility function is symmetric. Therefore, the consumption of both goods is increasing at all prices. For Leontief preferences, the LSE property is established below by a geometric argument. This implies that good 1 is increasing. The same comparative static result cannot be established by the traditional method, which relies on the implicit function theorem, since U is not differentiable.

Example 15 *Suppose that* $U(x) = \min\{x_1/a_1, x_2/a_2\}$*. Take two incomparable points* x *and* y*, with* $x_1 \geq y_1$*. Suppose that* x *lies on a (weakly) higher indifference curve than* $x \wedge y$*. Then* x *lies either on a horizontal or on a vertical segment of some indifference curve. If* x *lies on a vertical segment, then both* $x \wedge y$ *and* y *lie to the left on the same vertical segment of a lower indifference curve, while* $x \vee y$ *lies in the same indifference curve as* x*. In that case,* $U(x \vee y) = U(x) > U(x \wedge y) = U(y)$*. Alternatively,* x *lies on a horizontal segment. In that case, both* $x \wedge y$ *and* y *lie to the left on the same vertical segment of a (weakly) lower indifference curve, and* $x \vee y$ *lies on a higher indifference curve than* x*. In this case,* $U(x \vee y) > U(x) \geq U(x \wedge y) = U(y)$*. With regard to the other implication, although* $U(y) \geq U(x \wedge y)$ *always since* y *always lies vertically above* $x \wedge y$*, for the same reason* $U(x \vee y) \geq U(x)$*. Therefore, the LSE*

[15] We focus here on the implications with weak inequalities, the case of the strict inequalities being similar.

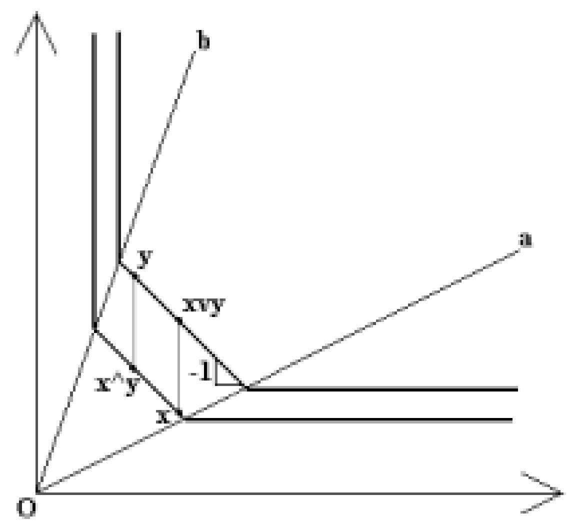

Figure 3:

conditions of Proposition 6 are satisfied so U is LSE in the direct value order, for all p.

The utility function of example 16, depicted in Figure 3, is a generalization of Leontief preferences.

Example 16 *Suppose that* $U(x) = \max_{\theta\in[0,1]} \min\left\{\frac{x_1}{\theta a_1+(1-\theta)b_1}, \frac{x_2}{\theta a_2+(1-\theta)b_2}\right\}$. *U is LSE in the direct value order, for all p.*

In these examples, the utility function is LSE in the direct value order $\geq_{d.v.(p)}$ for any given slope p. Since high and low budget sets are ranked by the strong set order when the direct value order is imposed on $\Re^2_+$, (V) implies that the set of optimizers is increasing with income in the strong set order. Therefore, the optimal choice(s) of good 1 are increasing in income (although the optimal choice(s) of good 2 need not be).

A noteworthy situation arises in the case of Example 16 when the relative price is $p = 1$, because $U(x)$ is not strictly quasiconcave. In that case, all points on the diagonal segment of the indifference curve are maximizers at some level of income, and the income expansion "path" is the area bounded by the rays Oa and Ob in Figure 3. The figure also depicts a selection of optimal points $\{x, y\}$ from the low and high budget sets, respectively, for $p = 1$. The two points are incomparable so the selection is not increasing in the direct value order. Furthermore, the choice of good 1 is decreasing over this selection. Yet the presumption is that good 1 should nevertheless be a normal good. Decreasing selections are a feature that is likely to arise when the optimal sets are not singletons, which is the motivation behind the broadened definition of a normal good given earlier. If need be, such nonincreasing selections can be ruled out by strengthening the theorem. This is done by requiring that the function be strictly superextremal rather than LSE (the details are provided in the appendix).

These last three examples are quasiconcave and LSE in $\geq_{d.v.(p)}$. However, quasiconcavity and LSE are independent properties. Example 12 illustrates that a utility function can be quasiconcave, but it is not LSE in the order $\geq_{d..v.(p)}$, if $p \in [1/2, 1]$ (as for the converse, Example 18 in section 3.4 is a utility function that is LSE for some direct value orders but not quasiconcave). In order to see when the utility function in Example 12 is LSE, it is necessary to compare the slopes of the indifference curves when these are defined with the slope p associated with the direct value order. This example illustrates the geometry behind the LSE property when there are two goods.

First, suppose that $p < 1/2$, and take a point like y in Figure 4. The join $x \vee y$, defined by the direct value order, lies to the right of y along the dashed line which has slope $-p$, and is flatter than the indifference curve. The indifference curve through the join $x \vee y$ is steeper than this line. Therefore, proceeding leftward from $x \vee y$ along the indifference curve, one passes above y. Therefore, $U(x \vee y) > U(y)$ for such prices, as the indifference curves are always steeper than the lines determined by the value order and, thus, $U(x)$ is LSE for$p < \frac{1}{2}$. A similar argument shows that $U(x)$ is LSE for $p > 1$ (in that case, the indifference curves are always flatter than the lines of the direct value order).

On the other hand, suppose that $p = 3/4$ and take two incomparable points x' and y', as in the figure, to be the $x-$intercept and the $y-$intercept of the bold segment, respectively. When $p = 3/4$, the indifference curve through x' is steeper than the value order line that connects x' with $x' \wedge y'$, so $U(x') > U(x' \wedge y')$. Conversely, the indifference curve through y is flatter than the value order line that joins it to $x' \vee y'$. Therefore, $U(y') > U(x' \vee y')$, so $U(x)$ is not

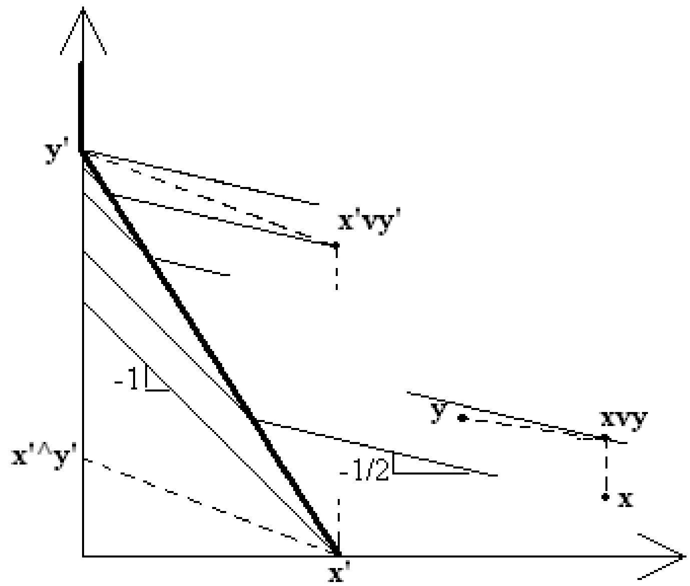

Figure 4:

LSE in the direct value order for that price. This should be expected, since good 1 is inferior at that price. The difference with the previous case is tied to the fact that between x' and y', the indifference curves go from being steeper than the value order line to being flatter than the value order line.

3.3 Normality for U Differentiable

If the utility function is twice differentiable, (V) leads to a known sufficient condition for good 1, which is that the marginal rate of substitution is increasing in good 2. The lattice-theoretic approach allows this result to be derived in a more general context than is customary. To see how the condition is derived, consider two incomparable points $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Without loss of generality, suppose that $y_1 = x_1 + i$ and that $y_2 = x_2 - \frac{p_1}{p_2}i - j$. Then, $x_1 \leq y_1$ and $p \cdot y = p \cdot x - j \leq p \cdot x$ so the points are indeed incomparable in the direct value order $\geq_{d.v.(p)}$. The meet and join of y and x are $x \wedge y = (x_1, x_2 - j)$ and $x \vee y = \left(x_1 + i, x_2 - \frac{p_1}{p_2}i\right)$ respectively.

The conditions for $U(x)$ to be LSE in the direct value order are

$$\begin{cases} U(x) = U(x_1, x_2) \underset{(>)}{\geq} U(x_1, x_2 - j) = U(x \wedge y) \\ \Longrightarrow U(x \vee y) = U\left(x_1 + i, x_2 - \frac{p_1}{p_2} j\right) \underset{(>)}{\geq} U(x_1 + i, x_2 - \frac{p_1}{p_2} j - i) = U(y), \text{ and} \\ U(y) = U(x_1 + i, x_2 - \frac{p_1}{p_2} i - j) \underset{(>)}{\geq} U(x_1, x_2 - j) = U(x \wedge y) \\ \Longrightarrow U(x \vee y) = U\left(x_1 + i, x_2 - \frac{p_1}{p_2} i\right) \underset{(>)}{\geq} U(x_1, x_2) = U(x) \end{cases}$$

The first of these implications holds directly since $U(x)$ is monotonic: y and $x \vee y$ correspond to an increase in the amount of good to relative to $x \wedge y$ and x respectively. Letting i go to zero, the second implication reduces to,

$$U_1(x_1, x_2 - j) - \frac{p_1}{p_2} U_2(x_1, x_2 - j) \geq 0 \Longrightarrow U_1(x_1, x_2) - \frac{p_1}{p_2} U_2(x_1, x_2) \geq 0.$$

This is a form of single-crossing property. It requires that the marginal rate of substitution MRS_{12}, viewed as a function of x_2, cross the relative price $\frac{p_1}{p_2}$ only once, from below, as x_2 increases.[16] Letting j go to zero, this last implication holds for all prices if the marginal rate of substitution is increasing in x_2, that is if $\frac{dMRS_{12}}{dx_2} \geq 0$ when $U(x)$ is twice differentiable.

The geometric interpretation of this sufficient condition is that the indifference curves become steeper as x_2 increases. It is the same condition that results from the usual, calculus-based, approach. However, there are differences inherent to the two approaches. First, the lattice-based approach does not assume that the optimum is interior. The sufficient condition therefore applies to any utility maximization problem for which $U(x)$ is monotonically increasing. A second, related aspect that will be discussed below is that $U(x)$ is not required to be quasiconcave.

A final fact is that LSE condition using direct value orders fully characterizes normality for all prices when the utility function is quasiconcave, in the sense that it provides a necessary and sufficient condition for normality.

Proposition 17 *Suppose that U is quasiconcave. Then, U is LSE in the order $\geq_{d.v.(p)}$ for all prices p if and only if good 1 is normal at all prices p for all levels of income.*

Proof. (V) establishes that LSE is sufficient for normality. For the reverse implication, if good 1 is normal at all prices, then indifference curves become

[16] In Figure 4 when the price is 3/4, the indifference curves cross the relative price in the wrong direction, in that they go from being steeper than 3/4 to being flatter as x_2 increases.

steeper as the amount of good 2 increases. Suppose that x and y are incomparable points with $x_1 \geq y_1$ and consider a given direct value order $\geq_{d.v.(p)}$. If $U(x) \geq U(x \wedge y)$, the indifference curve through x passes above $x \wedge y$. The indifference curve that passes through $x \vee y$ is everywhere steeper for a given amount of good 1, and therefore it must pass above y, so $U(x \vee y) \geq U(y)$. The same holds with strict inequalities, so U is LSE in the direct value order $\geq_{d.v.(p)}$ for all p. ■

3.4 The LSE Property when U is not Quasiconcave

Suppose that the lattice space is $\left[\Re^2_+, \geq_{d.v.(p)}\right]$, but now suppose that U is not quasiconcave. In this case, the expansion path may not be connected. It is still possible to verify whether U is LSE as in section 3.2., and (V) can be applied to conclude that the optimum set increases in the strong set order if U is LSE. However, the next example shows that (V) no longer produces a complete characterization of utility functions for which good 1 is normal. Take the preferences in Figure 5.

Example 18 *Suppose that* $U(x) = \begin{cases} x_1 + \frac{x_2}{2},\ x_1 \geq x_2 \\ \frac{x_1}{2} + x_2,\ x_1 \leq x_2 \end{cases}$. *$U$ is LSE in the direct value order for* $p \in [0, \frac{1}{2}) \cup (2, \infty)$, *but not for* $p \in \left[\frac{1}{2}, 2\right]$.

Theorem (V) establishes that the optimal choice of good 1 is increasing for $p \in [0, \frac{1}{2}) \cup (2, \infty)$ but does not apply to when $p \in \left[\frac{1}{2}, 2\right]$, even though the optimal choice(s) are increasing for $p \in \left[\frac{1}{2}, 2\right]$.

This example illustrates why (V) can generally be expected to provide only a sufficient condition for comparative statics in applications, even though the statement is an equivalence relation. Since $U(x)$ is not LSE for $p \in \left[\frac{1}{2}, 2\right]$, by (V) the optimal sets cannot be increasing over all pairs of strongly increasing constraint sets. There must be a pair of constraint sets that is ranked by the strong set order over which the optimum set is not increasing. Figure 5 depicts one such pair. The sets A and B are not the full budget lines that arise in economic applications, but are nevertheless ranked by the strong set order. A_2 is the optimal choice on the "low" set, while both B_1 and B_2 are optimal on the "high" set. On these sets, the optimizers are therefore not increasing in the strong set order even though B is larger than A. Formally, the theorem correctly captures a lack of monotonicity in this case. Indeed, the optimal set

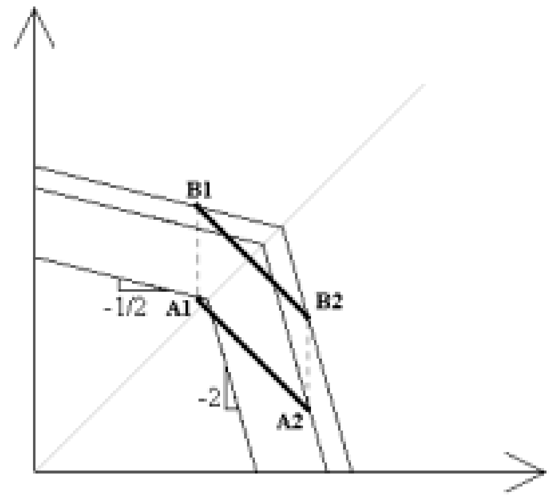

Figure 5:

is not increasing (in the strong set order). But this does not imply that the optimal set might not be increasing over all fully-ranged budget sets, or that good 1 is normal.

In the lattice space $\left[\Re^2_+, \geq_{d.v.(p)}\right]$, (V) therefore sometimes captures the fact that the optimum set increases as budget sets increase even though the utility function is not quasiconcave. In the next example, however, $U(x)$ is neither quasiconcave nor LSE for any direct value order, even though the optimum set increases with income for any price. These preferences are depicted in Figure 6.

Example 19 *Suppose that* $U(x) = \max\{\min\{x_1/a_1, x_2/a_2\}, \min\{x_1/b_1, x_2/b_2\}\}$. *$U$ is not LSE in the direct value order for any p. Take x and y as in Figure 6, for a given slope p, $x \wedge y$ must lie on a lower level set. This violates the strict implication in Proposition 6. Therefore, U is not LSE in the direct value order for any p.*

Since U is not LSE for any slope p, (V) cannot be applied. Nevertheless, the optimal set is increasing for any relative price. It is, depending on p, the ray Oa, or Ob, or both. As before, the theorem correctly reflects the fact that the optimal set does not increase over all increasing constraint sets,[17] even though

[17] Take $A = \{x, x \wedge y\}$ and $B = \{y, x \vee y\}$ with x and y as shown in Figure 6.

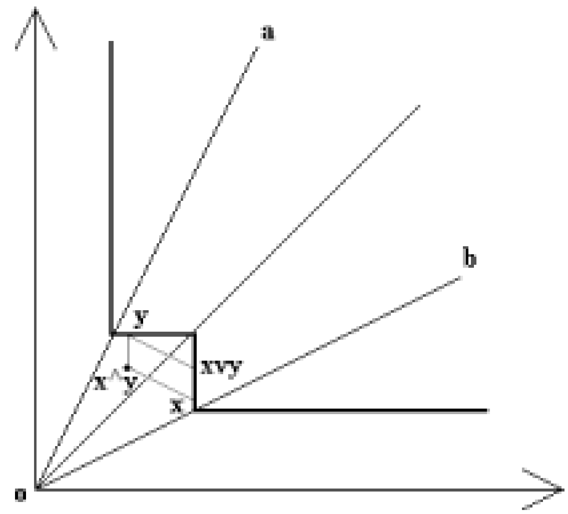

Figure 6:

it does for all budget sets.

The direct value order does not allow (V) to capture increasing expansion paths when U is not quasiconcave. However, there are other partial orders that can be imposed on $\Re_+^2$ that can be useful for such preferences. One example is the *radial value order* ($\geq_{r.v.(p)}$) which is given by:

Definition 20 $x \geq_{r.v.(p)} y$ *if* $x_2/x_1 \leq y_2/y_1$ *and* $p \cdot x \geq p \cdot y$

In geometric terms, x is greater than y with respect to this partial order if it lies on a higher budget set and on a ray from the origin with a smaller slope. A noteworthy feature of this partial order is that an increase with respect to $\geq_{r.v.(p)}$ reflects not only that good 1 is normal, but that its expenditure share is increasing. Figure 7 illustrates how the radial value order makes the utility function of Example 19 LSE (it also makes the utility function of Example 18 LSE, for all prices). Suppose that the lattice space is $\left[\Re_+^2, \geq_{r.v.(p)}\right]$ and take x and y so that $U(x) = U(x \wedge y)$. With the radial value order, both $x \wedge y$ and x lie on a line of slope $-p$, and both y and $x \vee y$ lie on a higher line of slope $-p$. Because $x \wedge y$ and y lie on one ray from the origin, and x and $x \vee y$ lie on another, $Oy/O(x \wedge y) = O(x \vee y)/Ox$, so $x \vee y$ and y must lie on the same

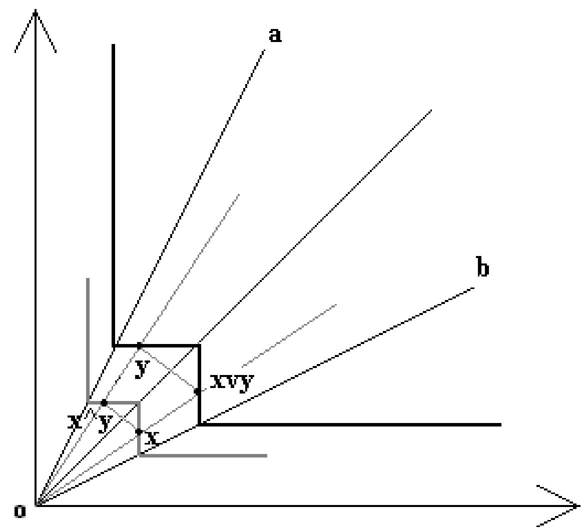

Figure 7:

indifference curve. Similar reasoning when $U(x) > U(x \vee y)$, establishes that U is LSE in the order $\geq_{r.v.(p)}$.

With the radial value order, (V) can be applied to some functions that are not quasiconcave but for which good 1 is normal. Therefore, the radial value order characterizes normality for some utility functions where the direct value order does not. Conversely. the bigger than sets of a given point in the radial value order are strict subsets of the corresponding bigger than sets in the direct value order of the same slope p. Therefore, there are cases where x_1 is normal and increases with respect to the direct value order, but not with respect to the radial value order. This happens if good 1 is normal but the upper bound of the expansion path is steeper than the ray from the origin at a given point (if the expenditure share on good 1 is increasing on a point of the upper bound of the constraint set). Therefore, the conditions that result from the use of the radial value order are neither stronger nor weaker than those that result from the use of the direct value order.

The radial order can be used to derive a sufficient condition for normality that has a geometric condition akin to that obtained in the case of the direct value order. Suppose that the utility function is twice differentiable. Consider two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ that are incomparable in the radial value order. Without loss of generality, take $p \cdot y = p \cdot x + j$ and $\frac{y_2}{y_1} = \frac{x_2}{x_1} + i$, so y is on a higher budget set than x, but on a steeper ray. Then, proceeding as in the case of the direct value order, a sufficient condition for $U(x)$ to be LSE in $\geq_{d.v.(p)}$ is,

$$\begin{aligned} &U_1(x_1, x_2) - \frac{p_1}{p_2} U_2(x_1, x_2) \geq 0 \\ &\qquad \Longrightarrow U_1(x_1 \frac{p \cdot x + j}{p \cdot x}, x_2 \frac{p \cdot x + j}{p \cdot x}) - \frac{p_1}{p_2} U_2(x_1 \frac{p \cdot x + j}{p \cdot x}, x_2 \frac{p \cdot x + j}{p \cdot x}) \geq 0. \end{aligned}$$

This holds if the marginal rate of substitution of x_2 for x_1 is increasing, not as x_2 increases as before, but along rays from the origin, that is if $\frac{dMRS_{12}}{dx_2}\Big|_{\frac{x_2}{x_1} = cst.} \geq 0$, when f is twice differentiable. The condition is satisfied by the preferences in figures 6 and 7, and by homothetic preferences in general.[18]

The classical calculus-based approach yields a similar sufficient condition.[19] As with the case of quasiconcave utility functions, the sufficient conditions that result from the lattice-based approach encompass the one that results from the classic calculus-based but eliminating the quasiconcavity restriction (which is effectively a second component of the classic sufficient condition), as well as the need for differentiability.

There are many partial orders that can induce lattice spaces over $\Re^2_+$ and that can capture normality. What is required is that the "level" sets which determine the order be upward sloping and, therefore, intersect the value lines only once. The radial value order illustrates that at least some of these orders may be economically pertinent, depending on the application. In the case of the last sufficient condition above, for example, the requirement of increasing marginal substitution along rays may be plausible if the goods considered are aggregates, in which case consumers can be interpreted to be choosing expenditure shares.

3.5 Income Effects with Arbitrarily Many Goods

The next example illustrates how (V) also applies to higher dimensional choice spaces including infinite-dimensional ones. Consider again the problem of find-

[18] The radial value order generalizes directly to induce lattice spaces over $\Re^n_+$, and similar sufficient conditions for normality therefore obtain in that setting.

[19] That is, take the first order condition $U_1(x, I - px) - pU_2(x, I - px) = 0$, and substitute in the income expenditure share $s = \frac{p_1 x}{I}$. $\frac{dx}{ds} \geq 0$, and $\frac{ds}{dI} \geq 0$ when $\frac{dMRS}{dx_2}\Big|_{\frac{x_2}{x_1} = cst.} \geq 0$, provided that U is quasiconcave.

ing sufficient conditions for good 1 to be normal, but suppose that there is an arbitrarily large, but countable, number of goods. It is first necessary to find a suitable generalization of the direct value order.

Suppose that an individual's preferences are represented by the utility function $U(x) : \times_{i=1,2,...}\Re_+ \longrightarrow \Re$. There are either N or countably many goods. Income is denoted by I and the prices of the goods are $p_1, p_2, \ldots$ Let $X = \times_{i=1,2,...}\Re_+$ be the choice space (in this application, the parameter enters naturally into the constraint set). The key step lies in generalizing the direct value order to higher dimensions.[20] This must be done in such a way that 1. $(X, \geq_X)$ is a lattice space, 2. budget sets are ranked by the induced strong set order $\geq_S$, and 3. an increase with respect to the partial order $\geq_X$ reflects an increase in x_1, corresponding to normality of good 1. Define the *canonical value order* $\geq_p$ as follows:

Definition 21 $x \geq_p y$ *if* $\sum_{i=1}^n p_i x_i \geq \sum_{i=1}^n p_j y_j$ *, for all* $n = 1, 2, ...$

With three goods, this signifies that $x \geq_p y$, if all of the following three weak inequalities are satisfied simultaneously,[21]

$$\begin{cases} p_1 x_1 \geq p_1 y_1, \\ p_1 x_1 + p_2 x_2 \geq p_1 y_1 + p_2 y_2, \\ p_1 x_1 + p_2 x_2 + p_3 x_3 \geq p_1 y_1 + p_2 y_2 + p_3 y_3. \end{cases}$$

The inequalities that characterize the canonical order are similar in the case of $N > 3$ goods.[22,23] Since $\geq_p$ is a partial order, $[X, \geq_p]$ is a lattice space. One feature that emerges from this definition is that the partial order $\geq_p$, and therefore

[20] Antoniadou [1995] uses a formulation that is based on a lexicographic ordering to define the direct value order. However, the joins that result from this order do not necessarily lie in the positive orthant when there are more than 2 dimensions.

[21] For example, suppose that $p = (1, 1, 1)$, and x and y are such that $x_1 \geq y_1$, $x_1 + x_2 \leq y_1 + y_2$ and $x_1 + x_2 + x_3 \leq y_1 + y_2 + y_3$. Then, $x \wedge y = (y_1, x_1 + x_2 - y_1, x_3)$ and $x \vee y = (x_1, y_1 + y_2 - x_1, y_3)$.

[22] It is clear that $\geq_p$ is reflexive as well as transitive. Antisymmetry holds by induction. If $x \geq_p y$ and $y \geq_p x$, then $x_1 = y_1$ by the first inequality, so $x_2 = y_2$, by the second (and so forth), so $x = y$.

[23] Suppose that there is a continuum of goods, which are indexed by a parameter $t \in A$. The extension of the canonical order is,

$$x \geq_p y \text{ if } \int_0^t p(s)x(s)ds \geq \int_0^t p(s)y(s)ds, \ \forall t \in A.$$

The $\geq_p$ relation is clearly reflexive and transitive. In order for antisymmetry to hold, it is necessary first to identify consumption streams that are equal almost everywhere, and also to restrict their value to be finite (suppose that the consumption set is a subset of X bounded above by the appropriate hyperplane), since two distinct consumption streams might have the same (infinite) value, even for some subset of the goods. As such, $\geq_p$ is a partial order on a relevant choice set. But with regards to characterizing income effects, it fails the third requirement mentioned in the text, namely, by identifying consumption streams that are equal a.e., it is not fine enough to distinguish increases in a single good such as x_1.

the lattice space $[X, \geq_p]$, are sensitive to the choice of indices, so that a change of indices alters the lattice space. The canonical order shares with the direct value order the property that budget sets with different incomes are ranked according to the induced strong set order. Suppose that $B = \{x \in X, \text{ s.t. } p \cdot x = \overline{I}\}$ and $A = \{x \in X, \text{ s.t. } p \cdot x = \underline{I}\}$ with $\overline{I} \geq \underline{I}$, and let $a \in A$ and $b \in B$. Then, $p \cdot (a \wedge b) = \underline{I} = p \cdot a$ (otherwise $a \wedge b \ngeq_p a$, violating the definition of the meet), so $a \wedge b \in A$ (and similarly $b \in B$). Therefore $B \geq_S A$.

This lattice space is characterized by its meet and join operations. Consider first the meet operation. In order to define the meet of two points x and y, it is necessary to introduce further notation. Denote the i-th component of $x \wedge y$ by $(x \wedge y)_i$, and let p^i, x^i, y^i, and $(x \wedge y)^i$ denote the price and quantity vectors corresponding to the first i components of p, x, y, and $x \wedge y$ respectively (for example $p^i = (p_1, p_2, \ldots, p_i)$). Since it is the greatest lower bound with regard to the partial order $\geq_p$, for a given $i > 1$, $x \wedge y$ satisfies,

$$\begin{aligned} p_i (x \wedge y)_i + p^{i-1} \cdot (x \wedge y)^{i-1} &= p^i \cdot (x \wedge y)^i \\ &= \left(p^i \cdot x^i\right) \wedge \left(p^i \cdot y^i\right), \end{aligned}$$

in order for the i-th inequality to hold (the value of the first i components of $x \wedge y$ cannot exceed that of the first components of x or of y).[24] Since $p^{i-1} \cdot (x \wedge y)^{i-1} = \left(p^{i-1} \cdot x^{i-1}\right) \wedge \left(p^{i-1} \cdot y^{i-1}\right)$,

$$p_i (x \wedge y)_i = \left(p^i \cdot x^i\right) \wedge \left(p^i \cdot y^i\right) - \left(p^{i-1} \cdot x^{i-1}\right) \wedge \left(p^{i-1} \cdot y^{i-1}\right),$$

which allows the meet of two points to be expressed recursively in terms on the "truncated values" such as $p^i \cdot x^i$. This is given in the following table

	$p^{i-1} \cdot x^{i-1} \geq p^{i-1} \cdot y^{i-1}$	$p^{i-1} \cdot x^{i-1} \leq p^{i-1} \cdot y^{i-1}$
$p^i \cdot x^i \geq p^i \cdot y^i$	$(x \wedge y)_i = y_i$	$(x \wedge y)_i = y_i + p^{i-1} \cdot \left(y^{i-1} - x^{i-1}\right) / p_i$
$p^i \cdot x^i \leq p^i \cdot y^i$	$(x \wedge y)_i = x_i + p^{i-1} \cdot \left(x^{i-1} - y^{i-1}\right) / p_i$	$(x \wedge y)_i = x_i$

The corresponding expressions for the components of the join of two points $x \vee y$ are derived in the same manner. The components of the joins and meets of the canonical order are such that $(x \wedge y)_i + (x \vee y)_i = x_i + y_i$ in all four of the above cases.

The expressions for the meet and join under the canonical order suffice to obtain a sufficient condition for all goods to be normal when the utility function

[24] The canonical order $\geq_p$ can be thought of as resulting from a change in variables. If one sets $\widetilde{x}(x) = (p_1 x_1, p_1 x_1 + p_2 x_2, \ldots)$, then $x \geq_p y$ if and only if $\widetilde{x} \geq_\varepsilon \widetilde{y}$.

is additively separable, i.e., $U(x) = \sum u_i(x_i)$. In the calculus-based approach, an analogous result can be obtained under the assumption that the component functions u_i are concave and differentiable, and that the optimum is interior. Proposition 2 does not even require these component functions to be continuous and rules out only interior satiation points (but not corners). This underscores the potential of the lattice-theoretic approach, on the one hand, as well as the restrictive nature of the separability assumption, which also lies behind the result, on the other.

Proposition 22 *If $U(x) = \sum u_i(x_i)$ and u_i is strictly monotone for all i, then all goods are normal.*

Proof. For U to be LSE, then,

$\sum u_i(x_i) \geq \sum u_i((x \wedge y)_i) \implies \sum u_i\left((x \vee y)_i\right) \geq \sum u_i(y_i)$, as well as the corresponding implication with strict inequalities for any two points x and y in X. We verify the LSE condition for a given i. Any of the four cases in the above table may hold for a particular $i > 1$. Proceed clockwise from the (top, left) case, where the condition reduces to $u_i(x_i) \geq u_i(y_i) \Longrightarrow u_i(x_i) \geq u_i(y_i)$. The inequalities that define the (top, right) case are such that $x_i \geq y_i$, necessarily. In this case, the i-th component of the meet has the form $(x \wedge y)_i = y_i + k$ and for the join $(x \vee y)_i = x_i - k$, with $k \geq 0$. U is, therefore, LSE if,

$$u_i(x_i) \geq u_i(y_i + k) \Longrightarrow u_i(x_i - k) \geq u_i(y_i).$$

Suppose that u_i is increasing (the decreasing case is analogous). If $k > x_i - y_i$, then $u_i(x_i) \ngeq u_i(y_i + k)$. If $k \leq x_i - y_i$, then $x_i - k \geq y_i$ so $u_i\left(x_i - k\right) \geq u_i(y_i)$. The restriction to strictly monotone functions in the proposition stems from the fact that otherwise, it may be that $u_i(x_i) > u_i(y_i + k)$ but $u_i(x_i - k) \not> u_i(y_i)$ which violates the strict inequality stipulation of the LSE condition. In the (bottom, right) case, the latter part of the LSE condition reduces to $u_i(y_i) = u_i(y_i)$, so the condition is satisfied trivially. Finally, in the (bottom, left) case $(x \wedge y)_i = x_i + k$ and $(x \vee y)_i = y_i - k$ with $k > 0$. Then, either the first inequality in the LSE condition isn't satisfied (if u_i is increasing) or the second one is (if u_i is decreasing) ■

If U is twice differentiable, then the canonical value order can be used to derive sufficient conditions for good 1 to be normal when there are arbitrarily many goods. These conditions generalize the one derived earlier in the two good setting. The three good case is given below.

Example 23 *For $N = 3$, good 1 is normal if the marginal rates of substitution satisfy,*

$$\begin{cases} \frac{dMRS_{12}}{dx_3} \geq 0, \ \frac{dMRS_{23}}{dx_3} \geq 0 \\ \frac{\frac{dMRS_{23}}{dx_1}}{\frac{dMRS_{23}}{dx_2}} MRS_{12} \geq 1, \ \frac{\frac{dMRS_{12}}{dx_2}}{\frac{dMRS_{12}}{dx_3}} MRS_{23} \geq 1. \end{cases}$$

An analogous set of conditions can be derived by altering the indices of goods 2 and 3 which alters the canonical value order and therefore the LSE conditions upon which the derivation is based. For N goods, one obtains, similarly, a set of 2^{N-1} conditions which have the same form as these.

These sufficient conditions are satisfied by commonly used functional forms, such as the CES. They differ from the sufficient condition that results from the classical calculus-based approach. They also differ from the condition that results from the calculus-based approach using the change in variables that is associated with the canonical value order (see footnote 20 above for another example of this method). This difference in the sufficient conditions may not be surprising since the lattice-based conditions do not require quasiconcavity in order to sign the comparative statics. Furthermore, both the lattice-based and the calculus-based approaches appear inherently different with more than two goods. The inequalities that emerge from the lattice-theoretic method are more numerous than the ones that emerge from the classic approach, but are also more simple to evaluate than the corresponding determinant conditions, in particular, as N gets large. Neither set of conditions implies the other so that lattice-based methods, besides placing fewer restrictions on the utility function in order to apply, impose different requirements as well.

4 Further Applications

In the previous section we illustrated how the choice of a partial order is instrumental in determining whether (V) applies to a given problem. Actually, in the case of income effects over two goods it is possible to use the Euclidean order rather than the direct value order, and still obtain the same sufficient conditions for income effects. Such a substitution may not always be fruitful but if it is, it is the easiest way to apply the lattice techniques. This was the approach taken in Amir, Mirman and Perkins, (1991).

Substituting the budget constraint into the objective function results in a reformulated maximization problem where the parameter enters the objective function. This yields the transformed utility function $\widetilde{U}(x_1, I) = U(x_1, I - px_1)$, in the two good case. With the constraint so integrated, the relevant lattice space is no longer defined over the consumption set $\Re^2_+$. Instead, we take $X = \left\{(x_1, I) \in \Re^2_+,\, I - px_1 \geq 0\right\}$, which corresponds to to the admissible consumptions of good 1 for any given income I. The difference between the previous formulation and this one is that X is defined both over a choice variable and the parameter I. Here, endowing X with the Euclidean order $\geq_\varepsilon$ is sufficient to make $[X, \geq_\varepsilon]$ a lattice space.[25] In order to study income effects, one takes the constraint sets to be $A = \{(x_1, I) \in X,\, I = \underline{I}\}$ and $B = \left\{(x_1, I) \in X,\, I = \overline{I}\right\}$. These sets are strongly ordered, so (V) applies. If $\widetilde{U}(x_1, I)$ is LSE in $\geq_\varepsilon$, then the set of optimum solutions is increasing with income. Let $x = \left(\underline{x}, \overline{I}\right)$ and $y = (\overline{x}, \underline{I})$ be two incomparable points in X. Their meet and join are $x \wedge y = (\underline{x}, \underline{I})$ and $x \vee y = \left(\overline{x}, \overline{I}\right)$ respectively. The conditions for U to be LSE are[26]

$$\left\{\begin{array}{c} U(x) = U\left(\underline{x}, \overline{I} - p\underline{x}\right) \geq U(\underline{x}, \underline{I} - p\underline{x}) = U(x \wedge y) \\ \Longrightarrow U(x \vee y) = U\left(\overline{x}, \overline{I} - p\overline{x}\right) \geq U\left(\overline{x}, \underline{I} - p\overline{x}\right) = U(y) \\ U(y) = U\left(\overline{x}, \underline{I} - p\overline{x}\right) \geq U(\underline{x}, \underline{I} - p\underline{x}) = U(x \wedge y) \\ \Longrightarrow U(x \vee y) = U\left(\overline{x}, \overline{I} - p\overline{x}\right) \geq U\left(\underline{x}, \overline{I} - p\underline{x}\right) = U(x) \end{array}\right.$$

The first of these implications holds by the monotonicity of U. The second one is, therefore, the more restrictive of the two. In the case where U is differentiable, it reduces to the single crossing condition,i.e., U_1/U_2 increasing in x_2.

[25] Let $x = (\overline{x}_1, \underline{I})$ and $y = (\underline{x}_1, \overline{I})$ be two incomparable points in X ($\overline{x}_1 \geq \underline{x}_1$ and $\overline{I} \geq \underline{I}$). Then $x \wedge y = (\underline{x}_1, \underline{I}) \in X$: since $x \in X$, $\underline{I} - p\overline{x}_1 \geq 0$. Similarly, $x \vee y \in X$.

With regard to the constraint sets A and B, $x \in A$ and $y \in B$, and that $x \wedge y \in A$ (similarly $x \vee y \in B$). Therefore, $B \geq_S A$.

[26] As well as the analogous conditions with strict inequalities.

In more general situations, the difficulty in applying (V) resides in determining an order so that the conditions of the theorem hold in the resulting lattice space, which is what the appropriate direct value order and the radial value order accomplish. In other applications, other kinds of difficulties can arise. One of these is that, unlike in the case of income changes, the parameter may not be distinct from the objective function.

4.1 Comparative Statics over Product Lattices

In this section we present two applications in which the parameter enters into the objective function. In these cases, an appropriate choice set X must be chosen and the maximization problem to which (V) is applied must also be formulated in a way that is consistent with the set X.

For the first example, suppose that the utility function is a parameter, and that there are two different types of consumers. A consumer's type can be represented simply by a scalar, such as the coefficient in a Cobb-Douglas utility function, but it can also be thought of as the utility function itself. (V) can be used to derive the result that a single-crossing condition is sufficient for the choices of different types facing the same budget constraint, to be ranked, without assuming that the utility functions are quasi-concave.

Let two individuals A and B, respectively, have the utility functions U^A and U^B defined over $\Re^2_+$. We find conditions on preferences that ensure that B's optimal choice of good 1 is greater than A's for any given level of income at prices p.

To apply (V) we first redefine the state space. Let $X = \Re^2_+ \times \left\{U^A, U^B\right\}$. The redefined objective function is,

$$\mathbf{F}\left(x, U\right) = U(x).$$

The constraint sets are $A = \left\{(x, U^A),\ x \in \Re^2_+ \text{ s.t. } p \cdot x = \underline{I}\right\}$ and $B = \left\{(x, U^B),\ x \in \Re^2_+ \text{ s.t. } p \cdot x = \overline{I}\right\}$.

Next, a partial order must be found to make X a lattice space. The first set in the product is endowed with the Euclidean order $\geq_\varepsilon$. With regard to utility functions, we impose $U^B \geq_U U^A$. For simplicity suppose that these utility functions are differentiable. In addition, suppose that they satisfy the single-crossing condition,

$$\frac{U_1^B}{U_2^B} \geq \frac{U_1^A}{U_2^A}, \text{ with } U_2^A, U_2^B \neq 0 \ \forall x \in \Re_2^+.$$

This single-crossing condition is illustrated in Figure 9. At an arbitrary point, the B type has a greater marginal rate of substitution that the A type. We take $U^B \geq_U U^A$ to mean that the single crossing condition is satisfied at all points. Thus, an "increase" in type is associated with steeper indifference curves by virtue of the single-crossing condition. Since, our purpose is to characterize a condition on utility functions that implies that the optimal set increases as the utility function U "increases", the partial order $\geq_U$ is determined in such a way as to make $\mathbf{F}$ LSE

We take the lattice space $[X, \geq_X]$ to be the "product lattice". Since X is the Cartesian product of two sets, $\geq_X$ is taken to be the component-wise partial order, i.e. $(x, U) \geq_X (y, V)$ if $x \geq_\varepsilon y$ and $U \geq_U V$. Assume that $x = \left(\overline{x}, U^A\right)$ and $y = \left(\underline{x}, U^B\right)$ are two incomparable points in X ($\overline{x} > \underline{x}$ and the utility functions are ordered as above). With the partial order $\geq_X$, the meet and join of x and y are $x \wedge y = \left(\underline{x}, U^A\right)$ and $x \vee y = \left(\overline{x}, U^B\right)$. A key feature of this product lattice is that it is composed of both the choice space and the parameter space. This allows the desired comparative static to be cast in terms of strongly increasing sets, by taking $A = \left\{(x, U^A),\ x \in \Re_+^2 \text{ s.t. } p \cdot x = I\right\}$ and $B = \left\{(x, U^B),\ x \in \Re_+^2 \text{ s.t. } p \cdot x = I\right\}$. Therefore, (V) applies and a sufficient condition for B's optimal choice set of good 1 to be greater than A's is that $\mathbf{F}$ be LSE in the partial order $\geq_X$.

Having defined a lattice space and determined the relevant constraint sets A and B, it remains to define the maximization problem formally. Set $\mathbf{F}(U, x) = U(x, \frac{I - p_1 x}{p_2})$ for $(x, U) \in X$. Then $\mathbf{F} : X \longrightarrow \Re$ is defined over the lattice space $[X, \geq_X]$.For a given utility function U, if x maximizes $\mathbf{F}(U, x)$, it maximizes $U(x)$ as well. This means that a maximum of $\mathbf{F}(U, x)$ when the domain of $\mathbf{F}$ is restricted to A (B) corresponds to an optimal consumption for a type A (B) individual. By (V), if $\mathbf{F}(U, x)$ is LSE then $B^* = \arg\max_{(U,x) \in B} \mathbf{F}(U, x) \geq_S \arg\max_{(U,x) \in A} \mathbf{F}(U, x) = A^*$, i.e.,The optimal consumption is increasing in type.

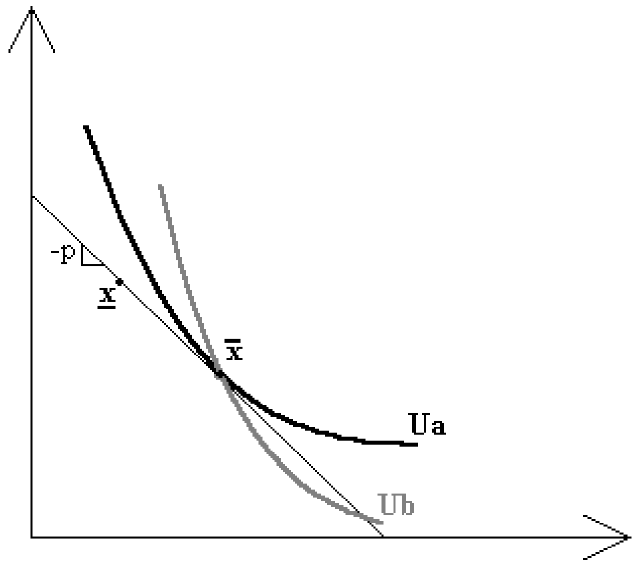

Figure 8:

Let $x = (\overline{x}, U^A)$ and $y = (\underline{x}, U^B)$ be two incomparable points. The conditions for F to be LSE are,[27]

$$\begin{cases} \mathbf{F}(x) = U^A(\overline{x}, \frac{I - p_1\overline{x}}{p_2}) \geq U^A(\underline{x}, \frac{I - p_1\underline{x}}{p_2}) = \mathbf{F}(x \wedge y) \\ \Longrightarrow \mathbf{F}(x \vee y) = U^B(\overline{x}, \frac{I - p_1\overline{x}}{p_2}) \geq U^B(\underline{x}, \frac{I - p_1\underline{x}}{p_2}) = \mathbf{F}(y) \\ \text{and} \\ \mathbf{F}(y) = U^B(\underline{x}, \frac{I - p_1\underline{x}}{p_2}) \geq U^A(\underline{x}, \frac{I - p_1\underline{x}}{p_2}) = \mathbf{F}(x \wedge y) \\ \Longrightarrow \mathbf{F}(x \vee y) = U^B(\overline{x}, \frac{I - p_1\overline{x}}{p_2}) \geq U^A(\overline{x}, \frac{I - p_1\overline{x}}{p_2}) = \mathbf{F}(x) \end{cases}$$

These sufficient conditions for F to be LSE do not require the differentiability of the utility function. However, when the utility functions are differentiable, the first of the LSE conditions reduces to

$$\frac{U_1^A}{U_2^A}(\underline{x}) \geq \frac{p_1}{p_2} \Longrightarrow \frac{U_1^B}{U_2^B}(\underline{x}) \geq \frac{p_1}{p_2}, \; \forall \underline{x} \in [0, I/p_1]$$

which holds by the single-crossing condition. Note that the second of the LSE conditions may fail to hold, given our assumptions. However, if the utility functions are ordinal, the single-crossing condition is sufficient for comparative statics, i.e., this condition can be satisfied by picking utility functions such that

[27] The reasoning with strict inequalities is analogous.

$U^A(x) > U^B(x)$, $\forall x \in [0, I/p_1]$ to represent the preferences of the two types. Then, the second LSE condition holds trivially.

Next, suppose that utilities take a quasilinear form,

$$U^i(x) = u^i(x_1) + x_2,\ i \in \{A, B\}\,.$$

Because the utility functions are quasilinear in x_2, the second LSE condition reduces to,

$$u^B(\underline{x}) \geq u^A(\underline{x}) \Longrightarrow u^B(\overline{x}) \geq u^A(\overline{x}),$$

which holds if,

$$u^B(\overline{x}) - u^B(\underline{x}) \geq u^A(\overline{x}) - u^A(\underline{x}).$$

Note that $U_2^A(\underline{x}) = U_2^B(\underline{x})$ and the differentiability of the utility functions then establish that this condition is also implied by the single-crossing condition.[28],[29]

As a second example, consider the case of an agent making choices under uncertainty and maximizing expected utility. When an individual makes choices under uncertainty, the objective function contains a density that can be thought of as a non-scalar parameter. One question that can be asked in this framework is under what conditions does an increase in the riskiness of the return on an asset lead to an increase in current consumption.[30] This application provides a further example of using product lattices to apply (V) and derive sufficient conditions for comparative statics.

Suppose that an individual with the utility function $u(c)$ where c denotes consumption, a discount rate β, and wealth w, chooses how much to invest in a risky asset. The return on the risky asset is randomly distributed with density

[28] In the continuum of types case with quasilinear utility $U(x) = u(x_1, \theta) + x_2$, these arguments establish that $u_{1\theta} \geq 0$ is sufficient for optimal choice to increase in type regardless of whether u is concave in x_1 or not.

[29] Suppose that the second l.s.e. condition with strict inequalities is violated:

$$U^B(\underline{x}, \frac{I - p_1\underline{x}}{p_2}) > U^A(\underline{x}, \frac{I - p_1\underline{x}}{p_2}) \text{ and } U^B(\overline{x}, \frac{I - p_1\overline{x}}{p_2}) < U^A(\overline{x}, \frac{I - p_1\overline{x}}{p_2})$$

Then, $A^* = \{(\underline{x}, U^B)\}$ and $B^* = \{(\overline{x}, U^A)\}$. Since $A^* \wedge B^* = \{(\underline{x}, U^A)\} \notin A^*$, it is not the case that $B^* \geq_S A^*$, so Theorem 9 is not applicable.

Milgrom and Shannon's formulation of theorem 9, i.e., MS, renders the second type of l.s.e. condition presented here immaterial.

[30] Athey [2002] discusses this example.

either $\underline{F}$ or $\overline{F}$, where $\overline{F}$ is "more risky" than $\underline{F}$, i.e. both densities have the same expectation and $\overline{F}$ stochastically dominates $\underline{F}$ in the second-order.

The individual's optimization problem is to choose current consumption c to maximize:

$$U(c) = u(c) + \beta \int u(s(w-c))dF(s)$$

The parameter of interest is the density F which enters into the objective function. Therefore we recast the problem over a product lattice X that composes the choice space with the parameter space. This allows the comparative statics to be formulated over constraint sets A and B that are subsets of the lattice X. Let $X = [0,w] \times \{\underline{F}, \overline{F}\}$. The order on $\Re^+$ is the customary one and the second component of this product is trivially endowed with a partial order by imposing $\overline{F} \geq \underline{F}$. The partial order on $(X, \geq_X)$, is taken to be the componentwise order. In this way, an increase with respect to the partial order $\geq_X$ corresponds to a simultaneous increase in current consumption and riskiness of the density. With this order, $(X, \geq_X)$ constitutes a lattice space.

Having defined the lattice space, take $A = [0,w] \times \{\underline{F}\}$ and $B = [0,w] \times \{\overline{F}\}$. Both of these are subsets of X, and thus $B \geq_S A$. The last step in applying (V) is to set up a suitable maximization problem over A and B. This is done by defining the "pseudo" objective function $\mathbf{F}(c,F) = u(c) + \beta \int u(s(w-c))dF(s)$. This function is defined over the lattice space $[X, \geq_X]$. Furthermore, its restrictions to the sets A and B reflect the underlying optimization problem. Formally, $A^* = \arg\max_{(c,F)\in A} \mathbf{F}(c,F) = \arg\max_{c\in[0,w]} u(c) + \beta \int u(s(w-c))d\underline{F}(s)$ (and similarly for B^*, $\overline{F}$). An optimum of $\mathbf{F}$ over A (B) corresponds to an optimum of $U(c)$ when the consumer faces the less risky (more risky) density.

By (V), if $\mathbf{F}(c,F)$ is LSE then $B^* \geq A^*$ so optimum choice(s) are increasing with riskiness in the strong set order sense. In order to verify that $\mathbf{F}(c,F)$ is LSE, there are two types of incomparable points at which the conditions need to be checked. Let $\overline{c} \geq \underline{c}$, and take $x = (\underline{c}, \overline{F})$ and $y = (\overline{c}, \underline{F})$ to be a pair of incomparable points. Given the partial order $\geq_X$ defined above, $x \wedge y = (\underline{c}, \underline{F})$ and $x \vee y = (\overline{c}, \overline{F})$. The LSE conditions are,[31]

$$\begin{cases} \mathbf{F}(\underline{c}, \overline{F}) \geq \mathbf{F}(\underline{c}, \underline{F}) \Longrightarrow \mathbf{F}(\overline{c}, \overline{F}) \geq \mathbf{F}(\overline{c}, \underline{F}) \\ \mathbf{F}(\overline{c}, \underline{F}) \geq \mathbf{F}(\underline{c}, \underline{F}) \Longrightarrow \mathbf{F}(\overline{c}, \overline{F}) \geq \mathbf{F}(\underline{c}, \overline{F}) \end{cases}.$$

[31] The reasoning with strict inequalities is analogous.

With regard to the first of these conditions, $\mathbf{F}\left(\underline{c}, \overline{F}\right) \geq \mathbf{F}(\underline{c}, \underline{F})$ corresponds to,

$$\int u(s(w-\underline{c}))d\overline{F}(s) \geq \int u(s(w-\underline{c}))d\underline{F}(s).$$

This is impossible if u is concave because $\overline{F}$ is more risky than $\underline{F}$. If u is convex, the inequality holds but the convexity implies that, $\mathbf{F}(\overline{c}, \overline{F}) \geq \mathbf{F}\left(\overline{c}, \underline{F}\right)$. To the extent that the concavity of u is a common assumption, the second of the two LSE conditions provides the more substantive restriction. It corresponds to,

$$\begin{aligned} u(\overline{c}) + \beta \int u(s(w-\overline{c}))d\underline{F}(s) &\geq u(\underline{c}) + \beta \int u(s(w-\underline{c}))d\underline{F}(s) \\ \Longrightarrow u(\overline{c}) + \beta \int u(s(w-\overline{c}))d\overline{F}(s) &\geq u(\underline{c}) + \beta \int u(s(w-\underline{c}))d\overline{F}(s). \end{aligned}$$

This constitutes a sufficient condition for first-period consumption to increase with riskiness. These inequalities can be rearranged and interpreted as follows. If the benefit from increasing current consumption is greater than the foregone expected benefit from investing in the low risk asset, the same must hold when the asset's distribution becomes more risky.

Although u has been assumed to be concave (or convex), no mention has yet been made of differentiability. A sufficient condition for $\mathbf{F}(c, F)$ to be LSE is that it be supermodular (see footnote 4 above). This occurs if,

$$\int \left[u(s(w-\underline{c})) - u(s(w-\overline{c}))\right] d\overline{F}(s) \geq \int \left[u(s(w-\underline{c})) - u(s(w-\overline{c}))\right] d\underline{F}(s).$$

If u is differentiable, this expression reduces to the condition,

$$\int su'(s(w-c))d\left(\overline{F} - \underline{F}\right) \geq 0,$$

which holds so long as $su'(s(w-c))$ is concave.[32] Conversely, if $su'(s(w-c))$ is convex, then current consumption decreases with riskiness and savings increase.

In section 3, the focus was to ensure that the choice set could be ordered to form a lattice space and it was necessary to use a non-euclidean order in order to ensure that changes in the parameter would be reflected by increases in strongly ordered sets. In the two previous applications it has not been necessary to incorporate the parameter into the lattice space in anything but the weakest

[32] This can be shown, for instance, for u thrice differentiable with a coefficient of relative risk aversion that is greater than 1 and increasing in wealth. Mirman [1971] compares the case with no uncertainty to the case with random returns and finds similar conditions to the ones derived here.

sense in order for the product lattice X to be well-defined and for (V) to be usefully applied. Comparative statics only involve a pairwise comparison of optimum sets. Therefore, it is only necessary to order a pair of constraint sets at a time. In a product lattice, these can be arbitrarily ranked as "high" and "low", regardless of whether the underlying parameter space has a more natural lattice structure. In these two applications, it is natural to think of the parameter as being discrete in this way.

5 Appendix

5.1 Strong Normality

The definition of normality in section 3 does not rule out the existence of non-increasing selections from the optimum set as income increases. The points can be incomparable, in which case some possible choices of good 1 decrease with income. In some applications it may be desirable to rule this type of case out. Consider the following alternative definition of a normal good:

Definition 24 *A good is strongly normal if every optimal choice at high income is weakly greater than every optimal choice at low income.*

This definition allows for the existence of multiple optimum solutions, but requires points in a selection from the optimum sets at low and high income to be comparable. Strong normality does not imply that optimal choices are strictly increasing, only that they are nondecreasing.

The formal apparatus of section 2 can be amended to produce sufficient conditions for strong rather than weak normality. First, the notion of a strictly superextremal (SSE) function must be introduced. The analogous characterization to Proposition 7 is:

Proposition 25 *(Veinott) f is SSE if and only if $f(x) \geq f(x \wedge y) \Rightarrow f(y) > f(x \vee y)$, or equivalently if and only if $f(x) < f(x \vee y)$ or $f(y) < f(x \wedge y)$, for all incomparable x and y in X.*

In addition, the notion of an increasing set must be amended. Set B is said to be *strongly greater than* set A ($B \geq_{St} A$) if:

Definition 26 *$B \geq_{St} A$ if $a \in A$ and $b \in B$ imply that $b \geq_X a$.*

We can now state the analog to (V).

Theorem 27 *(Veinott) $f : [X, \geq_X] \to \Re$ is strictly superextremal if and only if $\arg\max_{x \in B} f(x) \geq_{St} \arg\max_{x \in A} f(x)$ for all A, $B \subseteq X$ with $B \geq_S A$, and so long as $\arg\max_{x \in A} f(x)$,$\arg\max_{x \in B} f(x) \neq \emptyset$.*

In Theorem 26, the stricter strong set order $\geq_{St}$ applies only to the comparison between the sets of optimizers, whereas the relationship demanded between the constraint sets A and B is the strong set order $\geq_S$ as in (V). Therefore, Theorem 26 applies in exactly the same lattice spaces and to the same constraint sets as (V). The stronger comparative statics statement results from the fact that the SSE property is more restrictive than the LSE property.

Consider Example 15 again. When the relative price is $p = 1$, all the points on the diagonal segment of an indifference curve are optimal for some level of income, so the expansion path is the entire area bounded by the rays Oa and Ob and has selections that are decreasing in x_1. We have seen that good 1 is normal in the weaker sense of section 3.

It can be shown that U is SSE in the direct value order when $p \neq 1$, so Theorem 26 implies that the optimum set is increasing in the order $\geq_{St}$ and good 1 is normal in the strong sense. When $p = 1$ it is possible to find incomparable points x and y so that $x \wedge y$ lies on the same indifference curve as x and $x \wedge y$ lies on the same indifference curve as y (see Figure 3). This violates the condition in Proposition 24, so U is not SSE at this price, and the optimal choice (set) is not strongly normal.

5.2 An Inferior Order?

Just as the use of different partial orders allowed (V) to be applied and characterize the normality of good 1 for different types of preferences, the use of a different partial order lets the theorem apply to inferior goods in some instances. To this end, Antoniadou [1995] proposes the *cross value order* which is defined as:

Definition 28 *$x \geq_{c.v.(p)} y$ if $y_1 \geq x_1$ and $p \cdot x \geq p \cdot y$*

This order is similar to the direct value order except that the inequality on the first component is reversed. It defines (weakly) bigger than sets that are depicted in Figure 10, and the resulting join and meet of two incomparable points are also shown. Their coordinates in the Euclidean order are $x \wedge y = (y_1, x_2 - p(y_1 - x_1))$ and $x \vee y = (x_1, y_2 + p\,(y_1 - x_1))$. As compared with the direct value order, an expansion path that lies in an optimal choice's bigger than set reflects a decrease in good 1 and is therefore consistent with good 1 being inferior.

Although the cross value order is suggestive, the lattice space that it induces is not useful in many economic applications. This is because the meets can lie

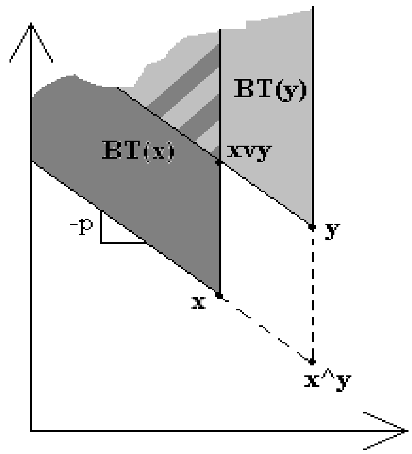

Figure 9:

below the horizontal axis, whereas utility functions are typically defined over the positive orthant. Therefore, (V) cannot be directly applied.

One solution to this problem is to consider utility functions that are defined for all $x \in \Re^2$ such that $x \geq_{c.v.(p)} 0$, i.e. for $x_1 \geq 0$ and $p \cdot x \geq 0$, though this may not always be justifiable on economic grounds. Alternatively, it may be possible to modify the cross value order so that the meets lie within the positive quadrant. There is an interesting asymmetry between normal and inferior goods beyond the consideration that not all goods can be inferior simultaneously. Normality can hold globally, in the sense that an income expansion path increase over the entire positive quadrant. Inferior goods on the other hand cannot be globally monotone unless the initial choice, and the choice at all higher incomes, is 0. Otherwise, they must be normal over some range before decreasing thereafter.

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